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1993 J. Phys. A: Math. Gen. 26 5923

(http://iopscience.iop.org/0305-4470/26/21/030)

J. Phys. A: Math. Gen. 26 (1993) 5923–5936. Printed in the UK

The spectrum of integral operators with logarithmically divergent norm

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Received 1 April 1993

Abstract. The spectrum and general characteristics of the solutions of non-Fredholm integral equations, which arise in the description of relativistic bound states, is investigated. An efficient method of numerical solution of such equations is suggested, which allows one to obtain accurate values of the solution for all physically relevant values of the coupling constant. The procedure is illustrated on specific examples.

1. Introduction

In the theoretical description of relativistic few-particle systems one frequently encounters integral equations with a kernel whose norm diverges logarithmically at large momenta. Such equations arise, for example, in various reductions of the Bethe–Salpeter equation (see, for example, [1–3] and citations therein) or in Hamiltonian variational approaches [4–6]. Similar equations arise also in various other problems in quantum field theory [7]. As a simplest example we might consider the so-called Salpeter equation [8]

$$(2\sqrt{p^2+M^2}-E)\psi(p)=\frac{f}{2\pi^2}\int\frac{d^3q}{(p-q)^2+m^2}\,\psi(q) \tag{1}$$

which arises in the description of two spinless particles of mass M, interacting via the exchange of a scalar quantum of mass m. Equation (1) is seen to be a relativistic momentum-space Schrödinger equation ($\hbar=c=1$) for the relative motion of the two-particles of mass M, with a Yukawa potential of range $r\simeq 1/m$ (note that $f/(Q^2+m^2)$ is just the Fourier transform of $f(e^{-mr}/r)$).

In the present article, we study the general behaviour of the eigenenergy spectra (for $E<2M$) and solutions of logarithmically singular equations of type (1), and consider efficient methods of numerical solution.

2. Reduction to Fredholm form

Let us consider spherically symmetric solutions of equation (1), since the nature of the large-momentum singularity does not differ in principle for higher partial waves. If we

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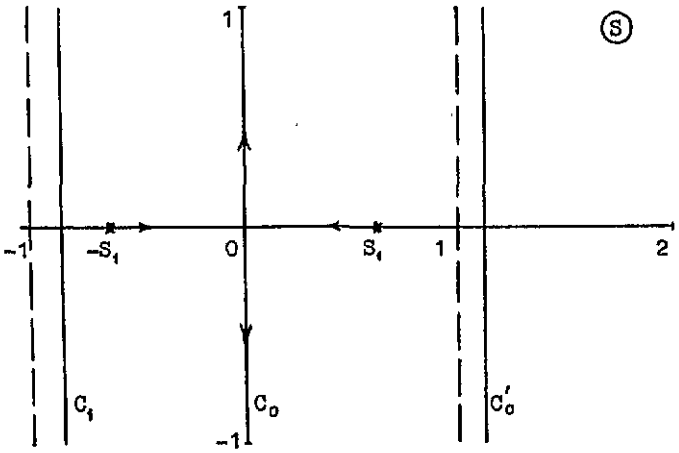

Figure 1. The complex s plane for equation (5), with $L(s)$ as given in equation (6). The motion of the roots s_1, as f increases, is indicated by the arrows.

make the substitution $\varphi(p)=p^2\psi(p)$ and measure all magnitudes in units of M, then $\varphi(p)$ satisfies the equation

$$\varphi(p)=\frac{p}{\sqrt{p^2+1}-E/2}\,\frac{f}{2\pi}\int_0^\infty \frac{\mathrm{d}q}{q}\,k(p,q,m)\varphi(q) \tag{2}$$

where

$$k(p,q,m)=\frac{1}{2}\ln\frac{(p+q)^2+m^2}{(p-q)^2+m^2} \tag{3}$$

and $E<2$. It is readily seen that (2) can be satisfied by the following asymptotic behaviour

$$\varphi(p)\sim p^{\pm s_1} \tag{4}$$

as $p\to\infty$, where the constants $\pm s_1$ depend only on the coupling strength f, and are roots of the equation

$$\mathcal{M}(s)=1-\tfrac{1}{4}fL(s)=0 \tag{5}$$

with

$$L(s)=\frac{1}{\pi}\int_0^\infty \mathrm{d}t\; t^{s-1}k(t,1,0)=\frac{2}{s}\tan\left(\frac{\pi s}{2}\right) \tag{6}$$

in the domain $-1<\mathrm{Re}\,s<1$. The roots $\pm s_1$ are close to ± 1 for small values of f, and converge to zero along the real axis as f increases towards $f_c=4/\pi$, beyond which these roots diverge along the Im s axis. This is illustrated in figure 1. If one of the constants which multiply the $p^{\pm s_1}$ terms is fixed by normalization, then the other remains unconstrained, and this suggests that equation (2) has a continuous spectrum of eigenvalues $E<2$, for given values of f and m.

To proceed with a more rigorous treatment, we restrict ourselves to the physically most relevant cases when $f<f_c$ and seek continuous solutions of equation (2) such that $\varphi(p)/p\to 0$, as $p\to\infty$. We analyse equation (2) by Mellin-transforming it to a singular

integral equation, using a procedure which was used previously, e.g. for investigating three-particle eigenspectra, with zero-range interactions, [9] and for analysing singular dispersion equations [10]. In the case of different fermion and scalar systems with massless particles ($M=m=0$) a similar technique leads to a finite difference homogeneous equations for the Mellin transforms, which have, though cumbersome, explicit solutions [11].

For values of p greater than an arbitrary constant λ, we rewrite (2) in the form

$$\varphi(p)=\frac{f}{2\pi}\int_{\lambda}^{\infty}\frac{dq}{q}\,k(p,\,q,\,0)\varphi(q)+\phi(p)\qquad p>\lambda \tag{7}$$

where $\phi(p)$ is the difference between the right-hand-side of (2) and the first term on the right of (7), and which we will formally regard as a given function. We introduce the dimensionless parameter $x=p/\lambda$ as well as the functions

$$\bar{\varphi}(x)=\varphi(\lambda x)\qquad \bar{\phi}(x)=\phi(\lambda x) \tag{8}$$

and their Mellin transforms

$$\tilde{\varphi}(s)=\int_{1}^{\infty}dx\;x^{s-1}\bar{\varphi}(x)\qquad \operatorname{Re}s>-1 \tag{9}$$

where, also

$$\bar{\varphi}(x)=\frac{1}{2\pi i}\int_{c_1}ds\;x^{-s}\tilde{\varphi}(s)\qquad x>1. \tag{10}$$

The contour c_1 in (10) must be chosen such that, in going from $-i\infty$ to $i\infty$, it passes to the left of all the singularities of $\tilde{\varphi}(s)$ (given that we are interested in the general solution), and in such a way that $\operatorname{Re}s>-1$. Note that the integral in (10) vanishes for $x<1$. Use of the above Mellin transforms allows us to rewrite (7) as the singular integral equation

$$\tilde{\varphi}(s)=\frac{1}{2\pi i}\int_{c_1}\frac{ds'}{s'-s}\,\frac{f}{4}\,L(s')\tilde{\varphi}(s')+\tilde{\phi}(s) \tag{11}$$

where s lies to the left of the contour c_1. The solutions of the singular integral equation (11) are characterized [12] by an index ν, which specifies the increase in the phase of the complex function $1/\mathcal{M}(s)$ as s changes along the contour of integration, namely

$$\nu=\frac{1}{2\pi}\operatorname{Im}\ln\frac{1}{\mathcal{M}(s)}\bigg|_{-i\infty}^{i\infty}. \tag{12}$$

The roots of the equation (5) lie on the real axis, at $\pm s_1, \pm s_2, \ldots$, where $0<s_1<1$, $2<s_2<3$, etc. Consider a contour c_1, as illustrated in figure 1, for which $-1<\operatorname{Re}s<-s_1$. It corresponds to the index $\nu(c_1)=1$. For such a contour the homogeneous equation, corresponding to (11), has a unique (canonical) solution which, for values of s that lie to the left of the contour c_1, can be written in the form

$$\tilde{\varphi}_0(s)=\frac{s-d}{s^2-s_1^2}\exp\left\{\frac{1}{2\pi i}\int_{c_1}\frac{ds'}{s'-s}\ln\left[\frac{s'^2-s_1^2}{s'^2-d^2}\,\frac{1}{\mathcal{M}(s')}\right]\right\} \tag{13}$$

where d is an arbitrary constant such that $\mathrm{Re}\, d > 1$. We note that the function $\tilde{\varphi}_0(s)$ is independent of the constant d. The general solution of (11) can be written as

$$\tilde{\varphi}(s) = A\tilde{\varphi}_0(s) + \tilde{\varphi}_p(s) \tag{14}$$

where A is an arbitrary constant and $\tilde{\varphi}_p(s)$ is a particular solution of (11), that is

$$\tilde{\varphi}_p(s) = \frac{\tilde{\varphi}_0(s)}{2\pi i} \int_{c_1} \frac{ds'}{s'-s} \frac{\frac{1}{4} fL(s')\tilde{\phi}(s')}{\mathcal{M}(s')\tilde{\varphi}_0(s')} + \tilde{\phi}(s). \tag{15}$$

Applying the inverse Mellin transform to (14), we obtain the result

$$\bar{\varphi}(x) = A\bar{\varphi}_0(x) + \bar{\varphi}_p(x) \qquad x > 1. \tag{16}$$

Recalling that, since the inhomogeneous term in (7) is, in turn, specified by the solution $\bar{\varphi}(x)$, $\bar{\varphi}_p(x)$ in (16) can be expressed as an ordinary one-dimensional integral of $\bar{\varphi}(x)$, upon interchanging the order of integration. If we now reinstate the variable $p = \lambda x$, and combine equation (16), for $p > \lambda$, with equation (2) for $p < \lambda$, we can replace (2) by the inhomogeneous integral equation

$$\varphi(p) = A\bar{\varphi}_0(p/\lambda) + \int_0^\infty dq\, K_1(p, q)\varphi(q) \tag{17}$$

which has a piecewise continuous kernel

$$\begin{aligned} K_1(p, q) &= \frac{1}{q} \frac{p^2}{\sqrt{p^2+1} - E/2} \frac{f}{2\pi} k(p, q, m) \qquad p < \lambda \\ &= \frac{1}{q} \frac{1}{2\pi i} \int_{c_1} ds\, p^{-s} T(q, s) \qquad p > \lambda \end{aligned} \tag{18}$$

where

$$T(q, s) = \frac{\tilde{\varphi}_0(s)}{2\pi i} \int_{c_1} \frac{ds'}{s'-s} \frac{\frac{1}{4} fL(s')R(q, s')}{\mathcal{M}(s')\tilde{\varphi}_0(s')} + R(q, s) \tag{19}$$

and

$$R(q, s) = \frac{f}{2\pi} \int_0^\infty dr \frac{r^s k(q, r, m)}{\sqrt{r^2+1} - E/2} - \theta(q-\lambda) \frac{f}{2\pi} \int_\lambda^\infty dr\, r^{s-1} k(q, r, 0) \tag{20}$$

with

$$\begin{aligned} \bar{\varphi}_0(x) &= \frac{1}{2\pi i} \int_{c_1} dx\, x^{-s} \tilde{\varphi}_0(s) \qquad \text{if } x > 1 \\ &= 0 \qquad \text{if } x < 1 \end{aligned} \tag{21}$$

and $\theta(p) = 0$ for $p < 0$ and $\theta(p) = 1$ for $p > 0$. The function $\bar{\varphi}_0(p/\lambda)$ in (17) is actually a solution of the homogeneous equation corresponding to (7).

It is convenient to express the meromorphic function $\mathcal{M}(s)$ of (5) as an infinite product of factors involving the poles and zeros of the function

$$\mathcal{M}(s)=\prod_{n=1}^{\infty}\left(\frac{s-s_n^2}{s-(2n-1)^2}\right). \tag{22}$$

On substituting this expression into (13), we obtain the result

$$\tilde{\varphi}_0(s)=\frac{1}{s+s_1}\prod_{n=1}^{\infty}\left(\frac{s-(2n-1)}{s-s_n}\right). \tag{23}$$

It then follows that

$$\tilde{\varphi}_0(x)=B_{-1}x^{s_1}+\sum_{n=1}^{\infty}B_n x^{-s_n} \qquad x>1 \tag{24}$$

$$B_{-1}=\prod_{k=1}^{\infty}\left(\frac{s_1+2k-1}{s_1+s_k}\right) \qquad B_n=\frac{s_n-(2n-1)}{s_n+s_1}\prod_{\substack{k=1\\k\neq n}}^{\infty}\left(\frac{s_n-(2k-1)}{s_n-s_k}\right) \tag{25}$$

and also

$$T(q,s)=R(q,s)/\mathcal{M}(s)+\tilde{\varphi}_0(s)P(q,s)$$

with

$$P(q,s)=\frac{1}{2\pi i}\int_{c_1^-}\frac{ds'}{s'-s}\,\frac{\frac{1}{4}fL(s')R(q,s')}{\mathcal{M}(s')\tilde{\varphi}_0(s')}. \tag{26}$$

The '$-$' superscript on c_1 is meant to indicate that s lies to the right of the contour c_1. We note that $P(q,s)$ has no poles in the domain $\operatorname{Re}s>-1$. Therefore $T(q,s)$ has poles at $-s_1, s_1, s_2, s_3, \ldots$ in the domain $\operatorname{Re}s>-1$, which orginate from $1/\mathcal{M}(s)$ and $\tilde{\varphi}_0(s)$, as well as poles at points $2, 3, \ldots$ which arise from $R(q,s)$ (the pole of $R(q,s)$ at $s=1$ is compensated by the zero of $1/\mathcal{M}(s)$). It then follows that

$$K_1(p,q)=p^{s_1}\theta(p-\lambda)h(q)/q+K_0(p,q) \tag{27}$$

with

$$h(q)=-R(q,-s_1)\ \text{Res}\ 1/\mathcal{M}(-s_1)-P(q,-s_1)\ \text{Res}\ \tilde{\varphi}_0(-s_1) \tag{28}$$

where the kernel $K_0(p,q)$ differs from $K_1(p,q)$ in that the contour c_1 is replaced by the contour c_0 in (18). The contour c_0 lies in the domain $-s_1<\operatorname{Re}s<s_1$ (in figure 1 c_0 is chosen to lie along the imaginary axis). The leading asymptotic behaviour of $K_1(p,q)$ is determined by the first term on the RHS of (27), that is $K_1(p,q)\sim p^{s_1}q^{-2}$ for $p,q\to\infty$. This means that the kernel $K_1^*(p,q)=\eta(p)K_1(p,q)\eta^{-1}(q)$ of the integral equation for the function $\varphi^*(p)=\eta(p)\varphi(p)$ is of finite norm, provided that $\eta(p)\sim p^{-1-s_1/2}$ for $p\to\infty$.

3. General characteristics of the solutions

As discussed above, the homogeneous equation (2), which is not of the Fredholm type, has been replaced by an inhomogenous Fredholm equation. Hence, equation (17) has a unique solution $\varphi(p)$ (which is independent of λ) for all $E<2$ which are not coincident with the eigenenergies of the integral operator with kernel $K_1(p,q)$. The arbitrary

constant A is fixed by normalization. The $p\to\infty$ behaviour of the solution, as generated by the relevant asymptotics of the inhomogeneous term and the kernel of (17), is

$$\varphi(p)=\alpha(p^{s_1}+\alpha_1 p^{s_1-1}+\alpha_2 p^{s_1-2})+\beta(p^{-s_1}+\beta_1 p^{-s_1-1})+O(p^{-2}). \tag{29}$$

where

$$\alpha_1=(E-fm)/(2\mathcal{M}(1-s_1)) \qquad \beta_1=(E-fm)/(2\mathcal{M}(1+s_1)) \tag{30}$$

and

$$\alpha_2=(E-fm+fm^2(1-s_1)\tan(\pi s_1/2)-1)/(2\mathcal{M}(2-s_1)). \tag{31}$$

The quantities $L(1+s_1)$, $L(2-s_1)$ that enter into the expression for β_1, α_2 are obtained by analytic continuation of (6) into the domain $\operatorname{Re} s>1$. One of the constants α or β may be taken to be arbitrary, but the quotient β/α is a given function of E which is determined by the detailed structure of the kernel of (2). The absence of a term of the type p^{-1} in (29) is a result of the fact that $T(q, s)$ has no pole at $s=1$. The terms of order $p^{\pm s_1-1}$, p^{s_1-2} arise from the expansion of $T(q, s)$ for $q\to\infty$. The terms with common α and β factors have been grouped separately in (29). If we arrange them in decreasing order as $p\to\infty$, we will obtain, for $s>\frac{1}{2}$, that is $0<f<1$, the sequence $p^{s_1}, p^{s_1-1}, p^{-s_1}$, $p^{s_1-2}, p^{-s_1-1}\ldots$. For the case $s<\frac{1}{2}$, that is $1<f<f_c$, the order of the sequence is different: the second and third terms must be interchanged, as must the fourth and fifth, and so forth.

The solution of the original equation (1) has, in general, the behaviour $\psi(p)\simeq \alpha p^{s_1-2}$ as $p\to\infty$. This is not a physically acceptable behaviour, since for example, the integral which defines the average kinetic energy

$$\int_0^\infty \mathrm{d}p\, p^2\sqrt{p^2+1}\,\psi^2(p)$$

diverges at the upper limit for all $s\in(0, 1)$. For values of $s\in(\frac{1}{2}, 1)$ even the normalization integral

$$\int_0^\infty \mathrm{d}p\, p^2\psi^2(p)$$

is divergent. Evidently, if $\psi(p)$ is to describe a bound state it must decrease faster than p^{-2} as $p\to\infty$.

Let us consider solutions of (2) such that $\varphi(p)\to 0$, when $p\to\infty$. In light of the above discussion, we must choose the contour c_0 rather than c_1 (figure 1) in the inverse Mellin transform of (10). With this choice of contour the index ν, which characterizes the singular integral equation (11), takes on the value $\nu(c_0)=0$. In that case, equation (11) has the unique solution

$$\tilde{\varphi}(s)=\tilde{\varphi}_p(s) \tag{32}$$

where $\tilde{\varphi}_p(s)$ is given by equation (15) but with the contour c_1 replaced by c_0. Performing the inverse Mellin transform we obtain, from equation (32), the result

$$\varphi(p)=\int_0^\infty \mathrm{d}q\, K_0(p, q)\varphi(q). \tag{33}$$

This Fredholm equation (33), with the kernel $K_0(p, q)$, which was obtained in section 2, can have a discrete spectrum of eigenvalues, E_i, in the domain $E<2$, since $K_0(p, q)$ is an analytic function of E in the domain of the semi-axis $E<2$. The eigenfunctions corresponding to E_i are simultaneously solutions of (17) with $A_i=0$ and

$$\int_0^\infty \frac{dp}{p} h(p)\varphi_i(p)=0. \tag{34}$$

In other words $\varphi_i(p)$ are also eigenfunctions of the integral operator with the kernel $K_1(p, q)$. The solutions of (2) which decrease as $p\to\infty$ have the behaviour $\varphi_i(p)\sim \beta_i p^{-s_1}$, with $\beta_i\neq 0$ in general. We now show that the solution $\varphi(p)$ of (2) becomes $\varphi_i(p)$ as $E\to E_i$. On multiplying equation (2) for $E=E_i$ and for $E\neq E_i$ by appropriate multipliers, subtracting and integrating from $p=0$ to $p=\lambda$, we obtain

$$(E-E_i)\int_0^\infty \frac{dp}{p^2}\varphi(p)\varphi_i(p)\,dp=\alpha\beta_i\frac{f}{\pi}\int_0^1\frac{dx}{x}\int_1^\infty\frac{dy}{y}k(x, y, 0)(x^{s_1}y^{-s_1}-x^{-s_1}y^{s_1}) \tag{35}$$

where we have set $p=\lambda x$ and taken $\lambda\to\infty$. The two-dimensional integral in (35) is not zero, hence the constant α, which appears as the coefficient of the first term in (29), goes to zero as $E\to E_i$, and so as $\varphi(p)\to\varphi_i(p)$. We stress that this fact is based totally on the asymptotic behaviour of the kernel of (2) at large momenta, and so it holds true for any other kernel that is different from this one only in the region of finite momenta. This shows, incidentally, that the eigenvalues E_i are non-degenerate. Equation (35) also implies that the solutions $\varphi(p)$ for $E\neq E_i$ are not orthogonal (with the weight function p^{-2}) to $\varphi_i(p)$, in contrast to solutions that correspond to different E_i. This is to be expected, since $\varphi(p)$ and $\varphi_i(p)$ satisfy different boundary conditions.

In summary, an equation of type (2), for given fixed values of m and $f\in(0, f_c)$, has a unique solution for every $E<2$ (except, possibly, in some special cases) with a continuous dependence on the energy. The high-p behaviour of the solution is given by equation (29). In the continuous spectrum $E<2$ there may exist values E_i for which the solutions behave as p^{-s_1} for $p\to\infty$, and for which $\alpha(E_i)=0$ and $\beta(E_i)\neq 0$. These particular values E_i correspond to eigenenergies of two-particle relativistic bound states of equation (1). The solutions which behave as p^{s_1} for $p\to\infty$ when $E\neq E_i$ are not physically relevant.

4. Reduced equation

The non-physical solutions, which grow as p^{s_1} for large p, disappear if a high-p cut-off is introduced into equation (2). Such a high-p cut-off may be physically motivated, for example by the finite size of the particles in question, etc. Let $E_i(\Lambda)$ and $\varphi_i(p, \Lambda)$ be the eigenvalue and corresponding eigensolution of (2) for the case where the upper limit on the integral is Λ. When $\Lambda\to\infty$, $E_i(\Lambda)$ approach E_i in accordance with

$$E_i(\Lambda)\simeq E_i-e_i\Lambda^{-2s_1} \tag{36}$$

where

$$e_i=\frac{f}{2\pi}\int_1^\infty\frac{dx}{x}\frac{\bar\varphi_0(x)}{B_{-1}}\int_0^1\frac{dy}{y}y^{s_1}k(x, y, 0)\left(\int_0^\infty\frac{dp}{p^2}\varphi_i^2(p)\right)^{-1} \tag{37}$$

and $\varphi_i(p)$ is a solution of (3) normalized such that $\varphi_i(p)=p^{-s_1}+\ldots$, for $p\to\infty$. Therefore the introduction of a high-momentum cut-off, Λ, into (2) allows one to extract

the discrete energies E_i from the $E<2$ continuum. In principle, this procedure can also be used to determine scattering solutions [13]. The direct numerical solution of (2) using quadrature formulas, automatically brings with it a cut-off procedure. However, as is evident from (36), the accuracy of such numerical solutions decreases as $f \to f_c$ since $s_1 \to 0$. This is evident, for example, from the results presented in [3] and [4]. In order to obtain results of a fixed accuracy, Λ would have to grow as $\exp\{(\text{const.})(f_c - f)^{-1/2}\}$ as $f \to f_c$. In the case of approximate variational solutions [4], the components for which $\varphi(p) \sim p^{s_1}$ as $p \to \infty$ are avoided by the appropriate choice of a normalizable trial function, and the accuracy of the solution depends entirely on the choice of trial wavefunction.

We now consider a different method of direct numerical determination of the eigensolutions. The solutions of (2) for $E = E_i$ have the behaviour ($\alpha = 0$, $\beta = 1$ in equation (29)), for $p \to \infty$:

$$\varphi_i(p) = p^{-s_1} + \beta_1 p^{-s_1 - 1} + Or(p^{-2}). \tag{38}$$

In light of the absence of a term of order p^{-1}, it follows from (2) that $\varphi_i(p)$ satisfies the condition

$$\int_0^\infty \mathrm{d}q(\varphi_i(q) - q^{-s_1}) = 0. \tag{39}$$

Thus we can separate out the slowly decreasing parts of $\varphi_i(p)$, by introducing the function

$$\chi_i(p) = \varphi_i(p) - \xi(p) \tag{40}$$

where

$$\xi(p) = \frac{p^2}{p^2+1} \frac{p+\beta_1}{p^{1+s_1}+1}. \tag{41}$$

Substituting (41) into (2) and using the condition (39), we obtain the following equation for the $\chi_i(p)$ function:

$$\chi(p) = \frac{p}{\sqrt{p^2+1} - E/2} \frac{f}{2\pi} \int_0^\infty \mathrm{d}q \left[\frac{1}{q} k(p, q, m) - k(p, 1, m)\right] \chi(q) + X(p) \tag{42}$$

where

$$X(p) = \frac{p}{\sqrt{p^2+1} - E/2} \frac{f}{2\pi} \int_0^\infty \mathrm{d}q \left[\frac{1}{q} k(p, q, m)\xi(q) - k(p, 1, m)(\xi(q) - q^{-s_1})\right] - \xi(p).$$

The inhomogeneous part $X(p)$ and the solution $\chi(p)$ of (42) behave as p^2 for $p \to 0$ and as p^{-2} for $p \to \infty$.

The analysis of the non-Fredholm type integral equation (42) is very similar to that discussed in the previous sections. The contour of integration in (11), which we denote by c_0', in this case lies in the domain $1 < \mathrm{Re}\, s < 2$ (figure 1), and $L(s)$ is taken to be the analytic continuation of the expression given in (6) into this domain. Since $\nu(c_0') = 0$, equation (11) has a unique solution which is given by (32). The inverse Mellin transformation changes (32) into an inhomogeneous Fredholm integral equation, where the inhomogeneity arises from $X(p)$. This means that (42) has a unique solution for all $E<2$, except for particular values of the energy for which the solution is obtained from

the homogeneous counterpart of (42). Those values of the energy $E=E_i$, at which the condition (39) is satisfied, correspond to the bound state eigenvalues of the system. The corresponding eigenfunctions are determined using (40) and (41), where

$$\chi_i(p)=\chi(p)|_{E=E_i}.$$

The principal advantage in the transformation of (2) to the form given by (39)–(42) is in the fact that $\chi(p)$ decreases sufficiently quickly at high momenta. The equations (42) and (39) can be solved by standard numerical methods. The effective introduction of a high-p cut-off, Λ, in (42) and (39) leads to an error in $E_i(f)$ which decreases as $1/\Lambda$ as $\Lambda\to\infty$ for all f in $(0, f_c)$. This implies the boundedness of the energy spectrum from below, since this is the case for finite Λ. The number of energy levels remains finite in the neighbourhood of $E=2$. This can be seen from the fact that the integral which determines the norm of the kernel remains convergent at the lower limit. In other words, equation (2) has a finite number of discrete eigenvalues, E_i.

5. Qualitative features of the solutions and examples

The positive definite nature of the kernel $k(p, q, m)$ in (2), that is,

$$\int_0^\infty \int_0^\infty \mathrm{d}p\ \mathrm{d}q\ k(p, q, m)\mu(p)\mu(q)>0$$

for any $\mu(p)$ implies that the values of E_i should decrease with increasing f or decreasing m. With decreasing m, the levels E_i appear at the point $E=2, f=f_c$ for some $m=m_{ci}$. Let us denote by f_{oi} the value of the coupling constant, which corresponds to the point where E_i begins to deviate from $E=2$. In that case we obtain the result

$$f_c-f_{oi}\simeq a_i(m_{ci}-m)^2 \qquad m\to m_{ci}. \tag{43}$$

For a given, fixed value of m, the behaviour of E_i is characterized by the following:

$$2-E_i\simeq b_i(f-f_{oi})^2 \qquad f\to f_{oi} \tag{44}$$

and

$$E_i-E_{ci}\simeq c_i(f_c-f)^{1/2} \qquad f\to f_c. \tag{45}$$

Since all magnitudes are measured in units of M, b_i in (44) and c_i in (45) are functions only of m/M, while a_i in (43) is a numerical constant.

The results given in equations (43)–(45) follow from the identity

$$0=\int \mathrm{d}p\left[\frac{1}{f_1}\left(2\sqrt{p^2+\frac{M_1^2}{m_1^2}}-\frac{E_1}{m_1}\right)-\frac{1}{f_2}\left(2\sqrt{p^2+\frac{M_2^2}{m_2^2}}-\frac{E_2}{m_2}\right)\right]\psi_1(m_1p)\psi_2(m_2p) \tag{46}$$

which holds for the solutions of (1) that correspond to different values of the parameters, and by using the asymptotic behaviour that $\psi(p)\sim[p^2+M(2M-E)]^{-1}$ when

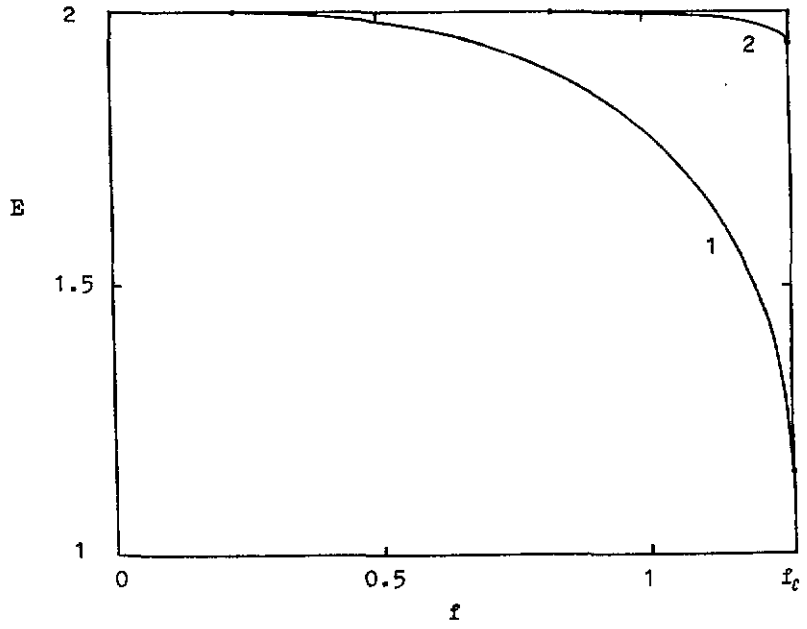

Figure 2. Eigenenergy spectrum of equation (2), for $m = 0.144$. The integers 1 and 2 denote the levels E_1 and E_2.

$E \to 2M$, $p \to 0$, and that $\psi(p) \sim p^{-2-s_1}$ when $p \to \infty$. Identity (46) is a consequence of the symmetry of the kernel $k(p, q, m)$ of (2). It follows from (46) that, for fixed m and M

$$\frac{\partial E_i}{\partial f} = -\frac{1}{f} \frac{\int \mathrm{d}\boldsymbol{p}\ (2\sqrt{p^2+1} - E_i)\psi_i^2(\boldsymbol{p})}{\int \mathrm{d}\boldsymbol{p}\ \psi_i^2(\boldsymbol{p})} < 0.$$

This means that the fact that E_i decreases with increasing f holds true even if $k(p, q, m)$ of (2) is not positive-definite, provided that it is symmetrical in p and q.

Figure 2 is a plot of solutions of equation (1) for $E_i(f)$ with the mass ratio (pion/nucleon) $m = 0.144$. These were obtained numerically by solving (42) and (39) using Gaussian quadrature formulae, having first evaluated the inhomogeneous term $X(p)$ to a sufficiently high accuracy. The system supports only two bound states, E_1 and E_2, which appear at $f_{01} = 0.24$ and $f_{02} = 0.83$ and descend to $E_{c1} = 1.15$ and $E_{c2} = 1.93$ as $f \to f_c$. In the $m \to 0$ limit, the asymptotic behaviour is of the form $f_{oi} \simeq m g_{ci}(1 + O(m^2))$, where g_{ci} is the critical constant of the corresponding non-relativistic problem, that is $g_{c1} = 1.68$, $g_{c2} = 6.2$, etc. When $m = 0$ there is an unlimited number of levels, which begin at $f = f_{oi} = 0$, and for small values of the coupling constant they have the behaviour

$$E_{nl} = 2 - \left(\frac{f}{2n}\right)^2 - \left(\frac{f}{2n}\right)^4 \left[\frac{n}{l+\frac{1}{2}} - \frac{3}{4}\right] + O(f^5) \tag{47}$$

where n is the principal and l is the orbital angular momentum quantum number. The result (47) is easily obtained using first-order perturbation theory with respect to the well known non-relativistic solutions.

It is of interest to note that for the case of two particles of distinct mass $M_1 \neq M_2$ and $M_1 = M$, $M_2 \to \infty$, when (1) reduces to an equation for a single particle moving in a Yukawa potential (with $E/2$ replaced by E and $f/2$ by f), the perturbative expansion

(47) coincides with that for the Klein-Gordon (KG) equation with a scalar Coulomb potential. The difference arises at order f^5, since for the KG case the expansion contains only even powers of f.

For $f \geqslant f_c$ the index, ν, of the singular integral equation (11) remains equal to unity. Therefore (2) has a unique solution for all $E<2$. Its behaviour, for $p \to \infty$, is characterized by the poles of the function $1/\mathcal{M}(s)$ in the domain $\operatorname{Re} s \in (-1, 1)$. For $f=f_c$ the function $1/\mathcal{M}(s)$ has a pole of order two at $s=0$:

$$\varphi(p) = \alpha \ln p + \beta + \ldots \qquad p \to \infty \tag{48}$$

For $f>f_c$ the poles at $\pm s$, diverge along the imaginary axis, that is

$$\varphi(p) = \gamma \sin(|s_1| \ln p + \delta) + \ldots \qquad p \to \infty. \tag{49}$$

This corresponds to the 'collapse to the centre' phenomenon, which also arises in non-relativistic problems with a $-g/r^2$, $g>\frac{1}{4}$, potential, or in the KG case with a Coulomb potential $-\alpha/r$ if $\alpha > \frac{1}{2}$. We note that for $f>f_c$ it is also possible to obtain a discrete spectrum, if the parameter δ in (49) is held fixed. The value of δ could be specified by considerations more general than those of equation (2), such as occurs in the analogous non-relativistic three-particle problem with zero-range interactions [9]. In other words the case $f>f_c$ requires a more elaborate determination of the interparticle interaction than that which is contained in (2).

The above considerations apply, in their general features, also for $l \neq 0$ partial waves, and for particles with spin. The corresponding equations differ from (2) in that the kernel, $k(p, q, m)$, is different and often more complicated, yet generally retains its logarithmically divergent character at the upper limit. Thus we investigated the equations obtained in [4] using the present technique. (These equations describe relativistic two-fermion bound states in QED with the inclusion of static Coulomb and transverse photon exchange interactions.) It turns out that the ${}^1S_0$, ${}^3P_0$, ${}^3S_1$, ${}^3P_1$ equations correspond to functions $L(s)$ for which the roots, $\pm s_1$, of (5) follow the trajectory of figure 1 as f increases. In other words these cases are totally analogous to that of (2), considered in the present work.

A quite different situation arises in the case of the ${}^1P_1$, equations A(1), A(7) and A(8) of [4b], for which the $L(s)$ function has the form

$$L(s) = -\frac{s}{4-s^2} \tan \frac{\pi s}{2} + \frac{3s}{1-s^2} \cot \frac{\pi s}{2}. \tag{50}$$

The character of the solutions of the equation for the ${}^1P_1$ states is determined by the roots of (5), with $L(s)$ as given in (50), in the domain $\operatorname{Re} s > -1$. Figure 3 illustrates the behaviour of the roots in the domain $\operatorname{Re} s \in (-1, 2)$ as well as those which appear here from the domain $\operatorname{Re} s < -1$ as f increases. The positions of the roots for particular values of the coupling constant f are denoted by the letters A, B, C, D and O. When $f=0$ there are four roots at the points A. As f increases they move along the real axis and when $f=f_1=1.0046$ they converge in pairs to the points B.

Thereafter the roots become complex and when $f=f_2=1.6105$ they cross the lines $\operatorname{Re} s = \pm 1$ at the points C. When $f=f_3=2.0918$ the roots come together in pairs at the points D. As f increases further, the roots move along the imaginary axis, two in opposite directions, and two towards each other. When f becomes equal to $f=f_4=2\pi/3=2.0944$ (this critical value was obtained in [4]), the roots arrive at the points denoted by O. As f increases still further, two roots continue to diverge along the imaginary axis, while the other two diverge from the origin of coordinates along the

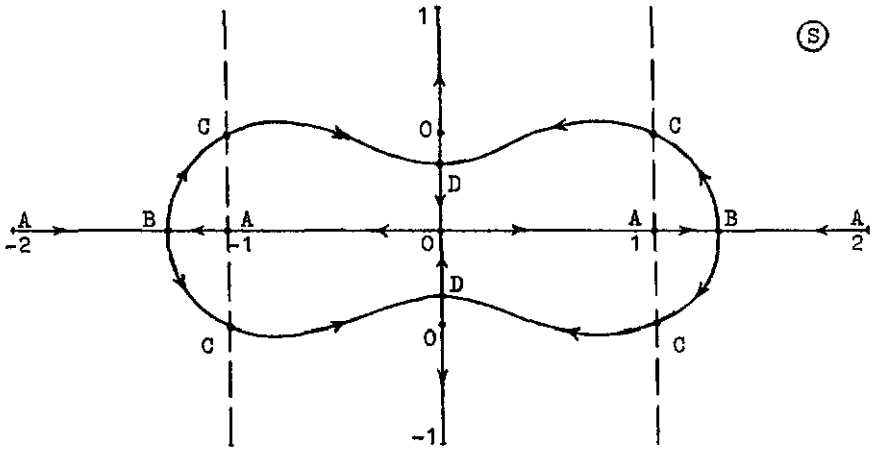

Figure 3. The complex s plane for equation (5) with $L(s)$ as given in equation (50). The letters A, B, C, D, O correspond to values of the coupling constant $f=0, f_1, f_2, f_3$ and f_4, respectively (see text). The arrows indicate the motion of the roots s_1, as f increases.

real axis. As $f\to\infty$ these roots head towards $\pm i\infty$ and ± 0.9188 respectively (these points are not marked in figure 3).

For $f<f_2$ the contour of integration, c_0, in the inverse Mellin transform (10), which lies in the domain $\mathrm{Re}\, s<-1$ to the left of all roots, is characterized by the index $\nu(c_o)=0$. This means that, for $f<f_2$, the equation can be reduced to a homogeneous Fredholm equation of the form (33) with a discrete eigenvalue spectrum. Furthermore, if $f<f_1$, the solutions behave as $\varphi_i(p)\sim p^{-s_1}$ when $p\to\infty$ (we have in mind here the smaller of the two roots s_1 in (1, 2)). However, if $f_1<f<f_2$ their behaviour becomes fundamentally different from the standard non-relativistic case: even for the ground state these solutions have an infinite number of nodes, corresponding to the asymptotic behaviour $\varphi(p)\sim p^{-s_1'}\sin(s_1''\ln p+\beta_1)$, where $s_1=s_1'+is_1''$. Correspondingly, the number of nodes is infinite in coordinate space at small distances. This anomaly is believed to be a manifestation, for sufficiently large values of f, of the main characteristics of the equation's kernel: in addition to the divergence of its norm as $p, q\to\infty$, the kernel is not sign-definite at large p, q with its sign being dependent on the ratio p/q. By contrast, the kernels for the ${}^1S_0$, ${}^3P_0$, ${}^3S_1$ and ${}^3P_1$ equations are positive-definite as $p, q\to\infty$, and the behaviour of the wavefunctions in these cases is consequently normal.

When $f>f_2$ two new roots from the domain $\mathrm{Re}\, s<-1$ appear in the domain $\mathrm{Re}\, s>-1$. The general solutions corresponds to the choice of contour c_2, which lies in the domain $\mathrm{Re}\, s>-1$ to the left of all roots, and has an index $\nu(c_2)=2$. This means that the ${}^1P_1$ equation, for $f>f_2$, can be written in the form of an inhomogeneous Fredholm equation, like (17), with an inhomogeneous part that depends on the two free parameters. In that case (for $f>f_2$) there are solutions for all values of the energy and they are characterized by the behaviour $\varphi(p)\sim Ap^{s_1'}\sin(s_1''\ln p+\alpha_1)+Bp^{-s_1'}\sin(s_1''\ln p+\beta_1)$ as $p\to\infty$. Purely decreasing solutions exist for particular values of the energy. They correspond to $A=0$, or to a choice of contour, c_o, which, for example, lies along the imaginary axis.

Among the continuum of solutions, for $f\geqslant f_3$, there are none that are decreasing as $p\to\infty$. Thus the domain $f\geqslant f_3$ corresponds to a continuous energy spectrum. When f increases beyond f_4 the behaviour of the solutions at large p changes substantially, but

the spectrum remains continuous. In short, the $^1P_1$ case is characterized by five critical values of the coupling constant, $f_0=0$ (at which the energy eigenlevels appear), and f_1 to f_4. Bound $^1P_1$ states exist in the domain $f_0<f<f_3$.

6. Parity doubling and mass degeneracy at critical coupling

Previous numerical solutions of relativistic two-particle bound state equations near the critical point $f=f_c$, beyond which the bound state eigenenergies $E_i(f_c)$ cease to be real [4b, 5, 14, 15], seem to suggest that the physically intriguing phenomenon of parity doubling and mass degeneracy of the eigenvalues, E_i, sets in. For example, for two-fermion systems like positronium e^+e^-, the J^P $0^-(^1S_0)$ and $0^+(^3P_0)$ states, which correspond to quite distinct energy eigenvalues at low coupling, seem to become degenerate as $f\to f_c$ [14, 15]. The same appears to occur for the $1^-(^3S_1)$ and $^1P_1$ levels. This degeneracy, as $f\to f_c$, might seem to be further widened to include unequal mass fermion systems such as μ^+e^-. Thus, the numerical solutions in that case [4b] may be interpreted as suggesting that there is also mass degeneracy at critical coupling, in that the E_i values, not only for opposite parities but also for different M_1/M_2 values, become degenerate as $f\to f_c$. Such interpretations, however, have to be checked carefully since all the previous numerical solutions referred to above suffer from a decrease in accuracy as $f\to f_c$.

However, the method discussed in the present paper allows us to make firmer conclusions about these supposed parity and mass degeneracies. To begin with, it is certainly true that the critical values, f_c, of the coupling constant exhibit parity and mass degeneracy. This is because the function $L(s)$ of (6) is determined only by the behaviour of the relevant equation's kernel at large momenta. Thus the functions $L(s)$ are identical for the $0^\pm(^1S_0$ and $^3P_0)$ states of a two-fermion system (cf equations (A1), (A3) and (A4) of [4b]):

$$L(s)=\frac{3}{s}\tan\left(\frac{\pi s}{2}\right)+\frac{s}{1-s^2}\cot\left(\frac{\pi s}{2}\right). \tag{51}$$

Moreover, equation (51) is valid for both the equal-mass (e^+e^-) and the unequal mass (μ^+e^-) two-fermion systems. Similarly, for the $1^\pm(^3S_1$ and $^3P_1)$ states (cf. equations (A1), (A5) and (A6) of [4b]) the function $L(s)$ is given by

$$L(s)=\frac{2(1-s^2)}{s(4-s^2)}\tan\left(\frac{\pi s}{2}\right)+\frac{2s}{1-s^2}\cot\left(\frac{\pi s}{2}\right). \tag{52}$$

This confirms the critical coupling values (when (5) with the relevant $L(s)$ has the root $s=0$), $f_c=8\pi/(4+3\pi^2)$ for the $0^\pm$ states, and $f_c=8\pi/(8+\pi^2)$ for the $1^\pm$ states, as well as the independence of f_c from the fermion masses, as was claimed in [4b].

On the other hand, physical quantities, including in particular energy eigenvalues E_i, are found to be both state and mass dependent as $f\to f_c$. For example calculations for the $0^\pm$ states, based on equations (A1), (A3) and (A4) of [4b] (which were transformed into the form, analogous to (39)–(42) of the present paper) resulted in different limiting eigenenergies, E_{ci}, for the $0^\pm$ states. Thus, for the case of equal masses, $M_1=M_2=M$ (positronium), these are found to be $E_c/M=1.2$ and 1.5 for the ground $0^-(^1S_0)$ and $0^+(^3P_0)$ states respectively. In particular we note that E_i does not descend to zero as $f\to f_c$, as might be suggested by the presence of the analytic solutions $\psi(p)=1/p^2$ when $E=M=0$ and $f=f_c$. (This shows that the $E, M\to 0$, and $M\neq 0, f\to f_c$ limits are

not equivalent.) Lastly we note that the symmetry of the equations' kernels for these cases leads to a relation of type (46) and, hence, to the decreasing character of the non-degenerate levels $E_i(f)$, with the limiting behaviour given by (45).

7. Concluding remarks

We have presented an analysis of the solutions of integral equations with a logarithmically divergent kernel, that arise in the relativistic description of two-particle bound states. The continuous solution spectrum is shown to possess a discrete set of normalizable eigensolutions at particular values of the energy, provided that the coupling constant is smaller than a particular critical value. We have shown how the equations can be transformed to a form that is particularly amenable to numerical solution by standard quadrature formulas for all relevant values of the coupling constant. We have illustrated this procedure on specific examples, including recent variational Hamiltonian treatments of two-fermion systems, such as e^+e^- or μ^+e^- [3, 4, 14, 15]. We also confirm the independence of the critical values of the coupling from the fermion masses and the parities of the states, but demonstrate that the eigenenergies do not exhibit such degeneracies at critical coupling.

Acknowledgments

This work was supported, in part, by the State Committee on Science and Technology of Ukraine. The first three authors acknowledge the support of the Suros Humanitarian Foundation under a grant awarded by the American Physical Society.

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